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THE 2-FOLD PURE EXTENSIONS NEED NOT SPLIT

A. A. Alijani

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Abstract. In this paper, we give an example of locally compact abelian groups A and C such that $\mathrm{Pext}^2(C, A) \neq 0$.

1. Introduction

Let $\pounds$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. The torsion subgroup and the first Ulm subgroup of $G \in \pounds$ are denoted by tG and G^1, respectively. A subgroup H of G is called pure if $nH = H \cap nG$ for all positive integers n. A morphism is called proper if it is open onto its image. An exact sequence $0 \to A \overset{\phi_n}{\to} B_n \to ... \overset{\phi_1}{\to} B_1 \overset{\phi_0}{\to} C \to 0$ in $\pounds$ is said to be an $n-$fold pure extension if each ϕ_i is a proper morphism and $\phi_i(B_{i+1})$ is pure in B_i where $B_{n+1} = A$ and $B_0 = C$. Following [**6**], we let $\mathrm{Pext}^n(C, A)$ denote the (discrete) group of $n-$fold pure extensions of A by C. In [**1**], an example of groups A and C in $\pounds$ was given such that $\mathrm{Pext}^2(C, A) \neq 0$. In the discussion of this example, the second part of [**1**, Proposition 8] is used, which is incorrect (see [**4**]). Let $B = \prod_p \mathbb{Z}(p)$, a product over all primes, considered as a discrete group. In this paper we prove that $\mathrm{Pext}^2(\mathbb{R}/\mathbb{Z}, tB) \neq 0$.

The additive topological group of real numbers is denoted by $\mathbb{R}$, $\mathbb{Z}$ is the group of integers and $\mathbb{Z}(n)$ is the cyclic group of order n. For any group G and H, $\mathrm{Hom}(G, H)$ is the group of all continuous homomorphisms from G to H, endowed with the compact-open topology. For more on locally compact abelian groups, see [**3**].

2. Preliminaries

An exact sequence $0 \to A \overset{\phi_n}{\to} B_n \to ... \overset{\phi_1}{\to} B_1 \overset{\phi_0}{\to} C \to 0$ in $\pounds$ is said to be an $n-$fold pure extension if each ϕ_i is a proper morphism and $\phi_i(B_{i+1})$ is pure in B_i where $B_{n+1} = A$ and $B_0 = C$ ([**6**]).

Theorem 2.1 ([**6**]). *The class* $\mathrm{Pext}^n(C, A)$ *of all equivalence classes of $n-$fold pure extensions of A by C forms a (discrete) group.*

The exact sequences (1) and (2) of the following proposition establish a closed connection between Hom and Pext in $\pounds$.

Proposition 2.2 ([**1**]). *Let $G \in \pounds$ and $0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ be a pure extension in $\pounds$. Then the following sequences are exact:*

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(1) $0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to \mathrm{Pext}(C,G) \to \mathrm{Pext}(B,G) \to \mathrm{Pext}(A,G) \to \mathrm{Pext}^2(C,G) \to ...$
(2) $0 \to \mathrm{Hom}(G,A) \to \mathrm{Hom}(G,B) \to \mathrm{Hom}(G,C) \to \mathrm{Pext}(G,A) \to \mathrm{Pext}(G,B) \to \mathrm{Pext}(G,C) \to \mathrm{Pext}^2(G,A) \to ...$

Recall that an LCA group is said to be have no small subgroups if there is a neighborhood of 0 which contains no nontrivial subgroups ([**7**]). Moskowitz proved that an LCA group G has no small subgroups if and only if $G \cong \mathbb{R}^n \oplus (\mathbb{R}/\mathbb{Z})^m \oplus D$ where n and m are nonnegative integers and D is a discrete group ([**7**, Theorem 2.4]).

Theorem 2.3 ([**5**, Theorem 2.4(iii)])**.** *Let A and C be groups in $\pounds$ such that A and C have no small subgroups. Then* $\mathrm{Pext}(C,A) = \mathrm{Ext}(C,A)^1$.

Proposition 2.4 ([**2**, Proposition 2.17(f)])**.** *Let D be a discrete group. Then* $\mathrm{Ext}(\mathbb{R}/\mathbb{Z}, D) \cong D$.

3. Main Result

Fulp [**1**] has concluded that $\mathrm{Pext}^2(G,\mathbb{Z}) \neq 0$ for some $G \in \pounds$. For proof of this claim, Fulp used the second part of [**1**, Proposition 8] which is incorrect, Since there exists a nonsplitting LCA group G whose torsion subgroup tG is finite ([**4**]). Hence, tG is an example of a discrete, reduced and algebraically compact group that is not pure injective in $\pounds$. In this section, we construct LCA groups G and H such that $\mathrm{Pext}^2(G,H) \neq 0$.

Theorem 3.1. *There exist groups G and H in $\pounds$ such that* $\mathrm{Pext}^2(G,H) \neq 0$.

Proof. Let $B = \prod_p \mathbb{Z}(p)$, a product over all primes, considered as a discrete group. We show that B/tB is divisible. Let $x = (x_p) \in B$ and $n \in \mathbb{N}$. If p is a prime where $(p,n) = 1$, let $y_p \in \mathbb{Z}(p)$ such that $ny_p = x_p$. Otherwise, let $y_p = 0$. Let $y = (y_p)$. Then $ny - x \in tB$. So B/tB is divisible. Now consider the pure exact sequence

$$0 \to tB \to B \to B/tB \to 0$$

By Proposition 2.2 , we have the following exact sequence

$$... \to \mathrm{Pext}(\mathbb{R}/\mathbb{Z}, B) \to \mathrm{Pext}(\mathbb{R}/\mathbb{Z}, B/tB) \to \mathrm{Pext}^2(\mathbb{R}/\mathbb{Z}, tB) \to ...$$

We first show that $\mathrm{Pext}(\mathbb{R}/\mathbb{Z}, B) = 0$. By Theorem 2.3 and Proposition 2.4,

$$\mathrm{Pext}(\mathbb{R}/\mathbb{Z}, B) = \bigcap_1^\infty n\mathrm{Ext}(\mathbb{R}/\mathbb{Z}, B) \cong \bigcap_1^\infty nB = B^1$$

.

Now, we show that $B^1 = 0$ (and hence $\mathrm{Pext}(\mathbb{R}/\mathbb{Z}, B) = 0$). Suppose $(x_p) \in B^1$. Then, $(x_p) \in nB$ for all positive integers n. Let p be an arbitrary prime number. Then, for every positive integer n, there is $y_p \in \mathbb{Z}(p)$ such that $ny_p = x_p$. Then, for $n = p$ we have $x_p = ny_p = 0$. Since

$$\mathrm{Pext}(\mathbb{R}/\mathbb{Z}, B/tB) = \mathrm{Ext}(\mathbb{R}/\mathbb{Z}, B/tB) \cong B/tB$$

and B is not torsion, it follows that $\mathrm{Pext}^2(\mathbb{R}/\mathbb{Z}, tB) \neq 0$. □

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A. A. Alijani
Department of Mathematics,
Technical and Vocational
University (TVU),
Tehran,
Iran
alijanialiakbar@gmail.com